On blocks estimators
for cluster inference of heavy-tailed time series

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Motivation
Goal

- Model extremal temporal clustering of stationary time series.
- Infer cluster statistics using block methods.

Key words: Extreme value theory, cluster Poisson point process, blocks methods for cluster inference, extremal index.
Table of contents

1. Cluster Poisson Process Q

2. Cluster inference

3. Example: causal linear model
We consider

- \((X_t)\) stationary time series in \((\mathbb{R}^d, |\cdot|)\).

\(^1\)Janssen (2019)
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- \((X_t)\) stationary time series in \((\mathbb{R}^d, |\cdot|)\).
- \((X_t)\) is regularly varying (RV\(\alpha\)): for all \(h \geq 0, y > 1,\)

\[
\lim_{x \to +\infty} \mathbb{P}(|X_0| > yx, \frac{X_{[\Theta]} \cap [-h,h]}{|X_0|} \in \cdot | |X_0| > x) = y^{-\alpha} \mathbb{P}(\Theta_{[-h,h]} \in \cdot).
\]

\(^1\text{Janssen (2019)}\)
We consider

- $(X_t)$ stationary time series in $(\mathbb{R}^d, |\cdot|)$.
- $(X_t)$ is regularly varying (RV$_\alpha$): for all $h \geq 0$, $y > 1$,

$$\lim_{x \to +\infty} \mathbb{P}(|X_0| > y x, \frac{X_{[-h,h]}}{|X_0|} \in \cdot | |X_0| > x) = y^{-\alpha} \mathbb{P}(\Theta_{[-h,h]} \in \cdot).$$

- $|\Theta_t| \to 0$ as $t \to +\infty$ a.s.
- $Q = \Theta / \|\Theta\|_\alpha$, where $\|x\|_\alpha = \sum_{t \in \mathbb{Z}} |x_t|^\alpha$. \(^1\)

\(^1\)Janssen (2019)
i.i.d. model

\( (X_t) \text{ i.i.d., } X_1 \text{ satisfies } RV_\alpha, \)

\[ Q_t = \Theta_t = \mathbb{1}(t = 0) \Theta_0. \]
Auto-regressive model

- $(X_t)$ a stationary $\textbf{AR}(1)$, $X_t = \varphi X_{t-1} + Z_t$ with $\varphi \in (0, 1)$, and $(Z_t)$ i.i.d. satisfying $\textbf{RV}_\alpha$,

\[ Q_t^{(\alpha)} = \frac{\Theta_t}{\|\Theta\|_\alpha} = \varphi^t \Theta^Z_0 \mathbb{1}(J + t \geq 0) (1 - \varphi^\alpha)^{1/\alpha}, \]

$J$ independent of $\Theta^Z_0$, $\mathbb{P}(J = j) = (1 - \varphi^\alpha) \varphi^{j\alpha}$, $j \geq 0$. 

![Graph of $Q_t^{(\alpha)}$]
Auto-regressive model

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Causal solution to SRE under Kesten-Goldie assumptions

- \((X_t)\) causal solution to SRE, \(X_t = A_tX_{t-1} + B_t\), \(((A_t, B_t))\) positive i.i.d. and \(((A, B))\) satisfies Kesten-Goldie theory then

\[
\Theta_t = A_t \cdots A_1, \quad t \geq 0,
\]

and \(\Theta_t \to 0\) a.s. since \(\mathbb{E}[\log(A_1)] < 0\) holds.

We take \(A_t = e^{N_t - 1/2}\) such that \((N_t)\) is i.i.d. gaussian noise, and we follow Example 6.1. in Janßen and Segers (2014) [5] where \(\Theta_{-t} = A_{-t} \cdots A_{-1}\), for \(t \leq 0\).
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Cluster Poisson Point Process

**Theorem** Buriticá, Meyer, Mikosch, Wintenberger (2021)

Assume \((X_t)\) satisfies \(RV_\alpha, AC\) and \(MX\), then

\[
N_n = \sum_{i=1}^{n} \varepsilon_a X_i \quad \rightarrow \quad N = \sum_{i=1}^{\infty} \sum_{j \in \mathbb{Z}} \varepsilon_{\Gamma_i^{-1/\alpha}} Q_{ij},
\]

in \(\mathcal{R}_0\), where \(nP(|X_1| > a_n) \rightarrow 1\),

- \(\sum_{j \in \mathbb{Z}} \varepsilon_{Q_{ij}}, i = 1, 2, \ldots\), is an iid sequence of point processes with state space \(\mathbb{R}^d\) with generic element \(Q_i = (Q_{ij})_{j \in \mathbb{Z}}\),

\[
Q = \left(\frac{\Theta_{ij}}{\|\Theta\|_\alpha}\right)_{j \in \mathbb{Z}}.
\]

- \((\Gamma_i)\) are points of a unit rate homogeneous Poisson process on \((0, \infty)\).
- \((\Gamma_i)\) and \((Q_i)_{i=1,2,\ldots}\) are independent.

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\(^2\)Davis and Hsing (1995); see [1, 3]
Cluster inference

Blocks method

**Aim:** compute cluster statistic: $\mathbb{E}[f(YQ)]$.

**So far:**

$N_n = \sum_{i=1}^{n} \varepsilon_{a_n^{-1}X_i} \rightarrow N = \sum_{i=1}^{\infty} \sum_{j \in \mathbb{Z}} \varepsilon_{\Gamma_i^{-1/\alpha}} Q_{ij}.$

$X_{[1:n]} = \left( X_{[1:b_n]} , X_{b_n+1:b_n} , \cdots , X_{[n-b_n+1:b_n]} \right),$ 

\[ := B_1 \]  
\[ := B_2 \]  
\[ := B_{mn} \]

- select $k$ extremal blocks $B_{(1)}, \ldots, B_{(k)}$,

- average $\frac{1}{k} \sum_{t=1}^{k} f(B_{(t)}/a_n)$,

- e.g. count threshold exceedances in a block 
  $f : (x_t) \mapsto \sum \mathbb{1}(|x_t| > 1).$
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$$N_n = \sum_{i=1}^{n} \varepsilon a_n^{-1} X_i \rightarrow N = \sum_{i=1}^{\infty} \sum_{j \in \mathbb{Z}} \varepsilon \Gamma_{i^{-1/\alpha}} Q_{ij}.$$

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- average $\frac{1}{k} \sum_{t=1}^{k} f(B(t)/a_n)$,
- e.g. count threshold exceedances in a block
  $f : (x_t) \mapsto \sum \mathbb{1}(|x_t| > 1)$.

(Q) How to choose those $k$ extremal blocks?
Large deviations of $\ell^p$-blocks

**Theorem**  Buriticá, Mikosch, Wintenberger (2023)

Assume $(X_t)$ satisfies $RV_\alpha$, and $(x_n)$ satisfies $AC(x_n)$, $CS_\alpha(x_n)$, and $nP(|X_1| > x_{b_n}) → 0$. Then,

$$nP(|X_{[1,b_n]}| > y x_{b_n}, \frac{x_{[1,b_n]}}{||x_{[1,b_n]}||_\alpha} \in \cdot \ | \ ||X_{[1,b_n]}||_\alpha > x_{b_n})$$

$$→ y^{-\alpha}nP(Q \in \cdot)$$
Large deviations of $\ell^p$-blocks

**Theorem**  Buriticá, Mikosch, Wintenberger (2023)

Assume $(X_t)$ satisfies $RV_\alpha$, and $(x_n)$ satisfies $AC(x_n)$, $CS_p(x_n)$, and $nP(|X_1| > x_{b_n}) \to 0$. Then, for $p > 0$,

$$P(\|X_{[1,b_n]}\|_p > y x_{b_n}, \frac{X_{[1,b_n]}}{\|X_{[1,b_n]}\|_p} \in \cdot | \|X_{[1,b_n]}\|_p > x_{b_n}) \to y^{-\alpha}P(Q(p) \in \cdot)$$

and

$$\lim_{n \to \infty} P(\|X_{[1,b_n]}\|_p > x_{b_n})/(b_nP(|X_0| > x_{b_n})) = c(p) = E[\|Q\|_p^\alpha],$$

$$\theta_{|X|} = c(\infty) \leq c(p) \leq c(\alpha) = 1, \text{ for } p \in (\alpha, \infty).$$
Blocks method

To infer $\mathbb{E}[f(YQ^{(p)})]$, 

$$
\widehat{f^Q}(p) := \frac{1}{k} \sum_{t=1}^{m} f(B_t/\|B_t\|_{p,(k+1)}) \mathbb{1}(\|B_t\|_p > \|B_t\|_{p,(k+1)}),
$$

The same quantity $f^Q$ can be estimated using different pairs $p', f'$ as

$$
f^Q = \mathbb{E}[f(YQ)] = \frac{\mathbb{E}[\|Q^{(p')}\|_p^{\alpha} f(YQ^{(p')}/\|Q^{(p')}\|_p)]}{\mathbb{E}[\|Q^{(p')}\|_p^{\alpha}]} \\
c(p') = \mathbb{E}[\|Q\|_{p'}^{\alpha}].
$$
Asymptotic normality

We propose to estimate the statistic $f^{Q}_{\alpha} = \mathbb{E}[f_{\alpha}(Y^{Q})]$ by

$$\widehat{f}^{Q}_{\alpha} := \frac{1}{k} \sum_{t=1}^{m} f_{\alpha}(B_{t}/\|B_{t}\|_{\alpha,(k+1)}) \mathbb{1}(\|B_{t}\|_{\alpha} > \|B_{t}\|_{\alpha,(k+1)})$$

**Theorem** Buriticá, Wintenberger (2024)

Under moment, bias and mixing conditions. There exists $k = k_{n} \to \infty$, $m/k \to \infty$, such that for suitable $f_{\alpha} : \ell^{\alpha} \to \mathbb{R}$.

$$\sqrt{k_{n}} \left( \widehat{f}^{Q}_{\alpha} - f^{Q}_{\alpha} \right) \overset{d}{\to} \mathcal{N}\left(0, \text{Var}(f_{\alpha}(Y^{Q})) + \kappa^{2}\sigma^{2}_{\alpha}\right), \quad n \to \infty,$$

and $k_{n}/k'_{n} \to \kappa$, with $\kappa \geq 0$, $Y$ independent of $Q$, and $\mathbb{P}(Y > y) = y^{-\alpha}$, for $y > 1.$
Tail index Hill estimator

\[
\frac{1}{\hat{\alpha}^n} := \frac{1}{\hat{\alpha}^n(k')} := \frac{1}{k'} \sum_{t=1}^n \log(|X_t|/|X|_{(k'+1)}),
\]

where \(|X|_{(1)} \geq |X|_{(2)} \geq \cdots \geq |X|_{(n)}\), and \(k' = k'(n)\) is a tuning sequence for the Hill estimator satisfying \(k' \rightarrow \infty\), \(n/k' \rightarrow \infty\), as \(n \rightarrow \infty\).
Remarks

- $\ell_\infty$-block methods for cluster inference studied in Drees and Rootzén (2010) [4] with high threshold $\chi : |X_t| > x_{b_n}$. We extend the analysis to $\ell_{\hat{\alpha}}$-cluster inference selecting the blocks whose $\ell_{\hat{\alpha}}$-norm exceed the high threshold $\chi$: $\|B_t\|_{\hat{\alpha}} > x_{b_n}$.
Remarks

- $\ell^\infty$-block methods for cluster inference studied in Drees and Rootzén (2010) [4] with high threshold $x : |X_t| > x_{b_n}$. We extend the analysis to $\ell^{\hat{\alpha}}$-cluster inference selecting the blocks whose $\ell^{\hat{\alpha}}$-norm exceed the high threshold $x$: $\|B_t\|_{\hat{\alpha}} > x_{b_n}$.

- We promote the use of order statistics of $\ell^{\hat{\alpha}}$-norm blocks such that

$$\|B\|_{\hat{\alpha},(k+1)/x_{b_n}} \xrightarrow{\mathbb{P}} 1.$$ 

where $k_n = \left\lceil m_n \mathbb{P}(\|B_{1,b_n}\|_{\alpha} > x_{b_n}) \right\rceil$. In this way $k_n$ points to the bias-variance trade-off in extreme value statistics.
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- $\ell^\infty$-block methods for cluster inference studied in Drees and Rootzén (2010) [4] with high threshold $x : |X_t| > x_{b_n}$. We extend the analysis to $\ell^{\hat{\alpha}}$-cluster inference selecting the blocks whose $\ell^{\hat{\alpha}}$-norm exceed the high threshold $x: \|B_t\|^{\hat{\alpha}} > x_{b_n}$.

- We promote the use of order statistics of $\ell^{\hat{\alpha}}$-norm blocks such that

$$\|B\|^{\hat{\alpha},(k+1)}/x_{b_n} \xrightarrow{\mathbb{P}} 1.$$ 

where $k_n = \lceil m_n \mathbb{P}(\|B_{1,b_n}\|^{\alpha} > x_{b_n}) \rceil$. In this way $k_n$ points to the bias-variance trade-off in extreme value statistics.

- It is common to take $k_n/k'_{n} \rightarrow 0$. In this case the asymptotic variance simplifies to $\text{Var}(f_{\alpha}(\mathbf{YQ}))$. 

Number of extreme blocks

Denote $k_n(p) = \lceil m_n \mathbb{P}(\| B_{1,b_n} \|_p > x_{b_n}) \rceil$ the extremal $\ell^p$-blocks, for a sequence of levels $(x_n)$ satisfying $\text{AC}, \text{CS}_p$.
For i.i.d. sequence $k_n = \lceil n \mathbb{P}(|X_0| > x_{b_n}) \rceil \sim k_n(\infty) \sim k_n(p) \sim k_n(\alpha)$ exceedances.

Heuristic on the number of extreme blocks:

$$k_n(p) \sim m_n \mathbb{P}(\| B_1 \|_p > x_{b_n}) \sim c(p) n \mathbb{P}(|X_0| > x_{b_n}) \sim c(p) k_n,$$

$$k_n(\alpha) \sim m_n \mathbb{P}(\| B_1 \|_\alpha > x_{b_n}) \sim n \mathbb{P}(|X_0| > x_{b_n}) \sim k_n,$$

Assuming also $\text{CS}_\alpha$, $\alpha$-cluster inference is justified. In this case the tuning parameter $k_n$ does not dependent on the underlying time dependencies.
Extremal index

Maximum domain of attraction
There exists \((a_n)\) such that

\[
\left( \mathbb{P}(|X_1| \leq xa_n) \right)^n \rightarrow G(x) := \mathbb{P}\left((\Gamma_1)^{-1/\alpha} \leq x\right), \quad n \rightarrow \infty,
\]

where \(G(x) = \exp\{-x^{-\alpha}\}\), for \(\alpha > 0, x > 1, n\mathbb{P}(X_1 > a_n) \rightarrow 1.\)
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There exists \((a_n)\) such that

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\]

where \(G(x) = \exp\{-x^{-\alpha}\}\), for \(\alpha > 0, x > 1\), \(n\mathbb{P}(X_1 > a_n) \rightarrow 1\).

(Leadbetter 1983) there exists \(\theta \in (0, 1]\) such that

\[
\mathbb{P}(\|X_{[1,n]}\|_\infty \leq xa_n) \rightarrow (G(x))^\theta, \quad n \rightarrow \infty.
\]
\[ \mathbb{P}(\|X_{[1,n]}\|_\infty \leq \times a_n) \to (G(x))^\theta, \quad n \to \infty. \]

\[ \implies (\mathbb{P}(X_1 \leq \times a_{b_n}))^{\theta_{b_n}} \sim \mathbb{P}(\|B_{1,b_n}\|_\infty \leq \times a_{b_n}) \sim G(x), \quad n \to \infty. \]

for \( x_{b_n} = \times a_{b_n} \) with \( \|B_{1,b_n}/x_{b_n}\|_\infty \to 0 \), as \( \log(1-x)/x \to 0 \) as \( x \to 0 \),

\[ \frac{\mathbb{P}(\|B_{1,b_n}\|_\infty > x_{b_n})}{b_n \mathbb{P}(|X_1| > x_{b_n})} \to \theta_{|X|}, \quad n \to \infty. \]

\[ \implies \text{Blocks estimator based in (Hsing 1991):} \]

\[ \hat{\theta}^B_{|X|} = \frac{1}{k_n b_n} \sum_{t=1}^{m_n} \mathbb{1}(\|B_t\|_\infty > |X|_{(k+1)}). \]
Example: extremal index

Cluster-based extremal index inference
For example, if \( f_\alpha : (x_t) \mapsto \| (x_t) \|_\infty^\alpha / \| (x_t) \|_\alpha^\alpha \), then,

\[
\theta_{|x|} = \mathbb{E}[\| Q \|^\alpha_\infty].
\]

\( \implies \) Estimator of the extremal index based on extremal \( \ell^\alpha \)-blocks.

\[
\hat{\theta}_{|x|} = \frac{1}{k} \sum_{t=1}^{m} \frac{\| B_t \|_{\hat{\alpha},\infty}}{\| B_t \|_{\hat{\alpha}}} \mathbb{1}(\| B_t \|_{\hat{\alpha}} > \| B \|_{\hat{\alpha},(k+1)}),
\]
Causal linear model

Theorem Buriticá, Wintenberger (2024)³

Let \((X_t) = \sum_{t \geq 0} \varphi_j Z_{t-j}\), such that \((Z_t)\) is i.i.d. and satisfies \(RV_\alpha\). For \(\rho > 0\), assume \(\varphi_t = O(t^{-\rho})\). Assume

1) \(f_p\) is bounded and \(\rho > 3 + 2/\alpha\).

2) there exist \(\kappa' > 0\), and \((k_n)\) satisfying \(k_n = O(n b_n^{-\kappa'-1})\),

Then, if a bias assumption holds and \(k/k' \to 0\),

\[
\sqrt{k}(\hat{f}_{\alpha}^Q - f_{\alpha}^Q) \overset{d}{\to} \mathcal{N}(0, \text{Var}(f_{\alpha}(YQ))), \quad n \to \infty.
\]

In particular, the \(\alpha\)-cluster based estimators for the extremal index has null asymptotic variance!

³see [2]
Implementation extremal index

If \( f_\alpha : (x_t) \mapsto \| (x_t)\|_\infty^\alpha / \| (x_t)\|_\alpha^\alpha \), then for \( p = \alpha \),

\[
\theta|_X = \mathbb{E}[\| Q \|_\infty^\alpha].
\]

\[\implies\] Estimator of the extremal index based on extremal \( \ell^\alpha \)-blocks.

\[
\hat{\theta}|_X = \frac{1}{k} \sum_{t=1}^{m} \frac{\| B_t \|_\infty^\hat{\alpha}}{\| B_t \|_\alpha^\hat{\alpha}} \mathbb{1}(\| B_t \|_\hat{\alpha} > \| B \|_\hat{\alpha}, (k+1)),
\]

For the autoregressive process AR(1): \( \nabla var(f_\alpha(Q)) = 0 \).
Implementation extremal index

Blocks estimator based in (Hsing 1991):

\[ \hat{\theta}^B_{|X|} = \frac{1}{k_n b_n} \sum_{t=1}^{m_n} \mathbb{1}(\|B_t\|_\infty > |X|_{(k+1)}). \]

Direct computations from Example 10.4.2 in (Kulik and Soulier 2020) yield

\[ \sqrt{k}(\hat{\theta}^B_{|X|} - \theta_{|X|}) \xrightarrow{d} \mathcal{N}(0, \sigma^2_\theta), \quad n \to \infty, \]

where \( \sigma^2_\theta \in [0, \infty) \), and

\[ \sigma^2_\theta = \theta^2_{|X|} \sum_{j \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \mathbb{E}[|Q_{j+t}^{(\alpha)}|^{\alpha} \land |Q_{t}^{(\alpha)}|^{\alpha}] - \theta_{|X|}. \]

For the autoregressive process AR(1): \( \sigma^2_\theta = 1 - \theta_{|X|} > 0. \)
Case study

Figure: Location of weather stations in France.
Take away

- How to choose extreme blocks plays an important role in inference.
- Estimation of the tail-index can help to stabilize the method.
- $\ell^{\hat{\alpha}}$-blocks inference yields robust result.
Questions?

Thank you for your attention!
Further perspectives

- How to define extremal directions of extremes in space and time?
- Different extremes episodes can have different causes, identify extreme with comparable features could help environmental scientists detect and characterize the natural phenomena leading to an extreme event.
References


Simulation setup

\[
\hat{\theta}_{|X|,\alpha} = k_n^{-1} \sum_{t=1}^{m_n} \frac{\|B_t\|_\alpha}{\|B_t\|_\alpha} \mathbb{1}(\|B_t\|_\alpha > \|B\|_\alpha, (k)),
\]

(1)

\[
\hat{\theta}^B_{|X|} = \frac{1}{k_n b_n} \sum_{t=1}^{m_n} \mathbb{1}(\|B_t\|_\infty > |X|_{(k+1)}).
\]

(2)

- We simulate 1000 AR(1) trajectories \((X_t)_{t=1,...,n}\), \(X_t = \varphi X_{t-1} + Z_t\), for \(n = 8000, 3000, 1000\).
- We fix \(k = k_n = n/b_n^2\) and we use that \(k_n(p) = o(n/b^{1+\kappa'})\),
- In this setting,

\[
0 = \text{Var}(f_\alpha(Y Q^{(\alpha)})) < \sigma^2 = 1 - \theta_{|X|}.
\]
Extremal index comparison

Figure: Boxplots based on 1 000 simulations of \((X_t)_{t=1,...,n}\) with \(n = 5 000\) for the estimation of \(\theta_{|X|} = 0.8\) in the \(AR(1)\) model with \(\varphi = 0.2\) and iid student(1) noise.